Exercises

"Multivariable Calculus with vectors" by Hartley Rogers, Jr.

p. 294 Q22, p. 295 Q24

Q22. The plane $x - y + 4z = 1$ intersects the elliptical cylinder given by the equation $2x^2 + 4y^2 = 3$. Find the points on the curve of intersection where the coordinate z has maximum and minimum values, and give these values. (Two constraint functions)

Solution:

The Objective function is $f(x,y,z) = z$

Constraint functions are

$$g(x,y,z) = x - y + 4z - 1 = 0$$

$$h(x,y,z) = 2x^2 + 4y^2 - 3 = 0$$

Apply Lagrange's multiplier method,

at a optimal point, $\exists\ \lambda, \mu \in \mathbb{R}$ s.t.

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \quad \text{———} \quad (1)$$

Direct computation gives

$$\nabla f(x,y,z) = (0,0,1)$$

$$\nabla g(x,y,z) = (1,-1,4)$$

$$\nabla h(x,y,z) = (4x, 8y, 0)$$

(Note that ∇g, ∇h are linearly independent except when $(x,y) = (0,0)$.)

$$(1) \Leftrightarrow \begin{cases} \lambda = \frac{1}{4} \\ \frac{1}{4} + 4\mu x = 0 \\ -\frac{1}{4} + 8\mu y = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \lambda = \frac{1}{4} \\ x = -\frac{1}{16\mu} \\ y = \frac{1}{32\mu} \end{cases}$$

By the constraint equation, $h(x,y,z) = 0$

$$2\left(-\frac{1}{16\mu}\right)^2 + 4\left(\frac{1}{32\mu}\right)^2 = 3$$

$$8 + 4 = 3 \cdot (32\mu)^2$$

$$\mu = \pm \frac{1}{16}$$

$(x, y) = (-1, \frac{1}{2})$ or $(1, -\frac{1}{2})$

Together with $g(x, y, z) = 0$.

When $(x, y) = (-1, \frac{1}{2})$, $z = \frac{1}{4}(1 - x + y) = \frac{5}{8}$

When $(x, y) = (1, -\frac{1}{2})$, $z = \frac{1}{4}(1 - x + y) = -\frac{1}{8}$

Another approach (The following is not very precise)

$$\begin{cases} x - y + 4z = 1 \\ 2x^2 + 4y^2 = 3 \end{cases}$$

$x - y + 4z = 1 \Rightarrow$ This gives you $z = z(x, y)$

$2x^2 + 4y^2 = 3 \Rightarrow$ This gives you $y = y(x)$

$\therefore$ You may take z as a function of x

This reduces to one-variable calculus.

From $x - y + 4z = 1$ ——— ②,

differentiate ② w.r.t. x, we have

$$1 + 4\frac{\partial z}{\partial x} = 0 \Leftrightarrow \frac{\partial z}{\partial x} = -\frac{1}{4}$$

differentiate ② w.r.t. y, we have

$$-1 + 4\frac{\partial z}{\partial y} = 0 \Leftrightarrow \frac{\partial z}{\partial y} = \frac{1}{4}$$

From $2x^2 + 4y^2 = 3$ ——— ③ ,

differentiate ③ w.r.t. x, we have

$$4x + 8y(x)\,y'(x) = 0$$

$$y'(x) = -\frac{x}{2y}$$

We define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = z(x, y(x))$$

Remarks :

I. $z(x,y)$ comes from Eq② ②

II. $y(x)$ comes from Eq② ③

III. $(x, y(x), z(x, y(x)))$ is a curve of intersection

Aim : Find the maximum and minimum value of f

$$0 = f'(x) = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot y'(x)$$

$$= -\frac{1}{4} + \frac{1}{4}\left(-\frac{x}{2y}\right)$$

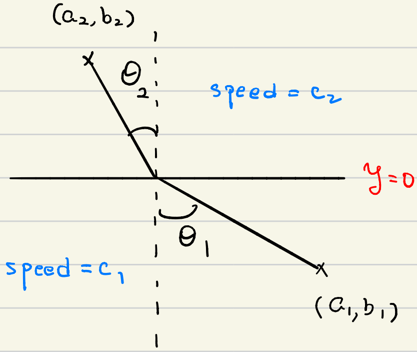
$$x = -2y$$

From eq^n ③, $$2x^2 + 4y^2 = 3$$
$$y^2 = \frac{1}{4}$$
$$y = \pm \frac{1}{2}$$

$\therefore$ $(x, y) = (-1, \frac{1}{2})$ or $(1, -\frac{1}{2})$. Same conclusion.

Q24. A light ray travels from point A to point B crossing a plane boundary between two different media. The path of the ray lies in a plane perpendicular to this boundary plane. In the first medium, its speed is c_1 and its direction makes an angle θ_1 with a line normal to the plane boundary. Similarly for c_2 and θ_2 in the second medium. Use a Lagrange multiplier to show that the trip is made in minimum time when Snell's Law holds:

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{c_1}{c_2}$$



Solution :

Minimize travel time

$$f(\theta_1, \theta_2) = \frac{-b_1}{c_1 \cos\theta_1} + \frac{b_2}{c_2 \cos\theta_2}$$

Subject to the constraint

$$-b_1 \tan\theta_1 + b_2 \tan\theta_2 = a_1 - a_2$$

Let $g(\theta_1, \theta_2) = -b_1 \tan\theta_1 + b_2 \tan\theta_2 - a_1 + a_2$

That is, $g(\theta_1, \theta_2) = 0$

Using Lagrange multiplier, aim to solve

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \left(-\frac{b_1}{c_1} \sec\theta_1 \tan\theta_1, \ \frac{b_2}{c_2} \sec\theta_2 \tan\theta_2\right) = \lambda\left(-b_1 \sec^2\theta_1, \ b_2 \sec^2\theta_2\right) & \text{—①} \\ g(\theta_1, \theta_2) = 0 & \text{—②} \end{cases}$$

$$① \Leftrightarrow \left(\frac{1}{c_1} \tan\theta_1, \ \frac{1}{c_2} \tan\theta_2\right) = \lambda\left(\sec\theta_1, \ \sec\theta_2\right)$$

$$\Leftrightarrow \lambda = \frac{1}{c_1} \frac{\tan\theta_1}{\sec\theta_1} = \frac{1}{c_2} \frac{\tan\theta_2}{\sec\theta_2}$$

$$\Leftrightarrow \frac{\sin\theta_1}{\sin\theta_2} = \frac{c_1}{c_2}$$

(Why this gives the minimum travel time?)

How about that?

Impose the condition that $0 \le \theta_1 < \frac{\pi}{2}$ and $0 \le \theta_2 < \frac{\pi}{2}$

Then, the boundary case is $\theta_1 = 0$ or $\theta_2 = 0$

However, it seems difficult to compute the value of $f(\theta_1, \theta_2)$ when θ_1, θ_2 satisfies both

$$\begin{cases} \dfrac{\sin\theta_1}{\sin\theta_2} = \dfrac{c_1}{c_2} \\ g(\theta_1, \theta_2) = 0 \end{cases}$$

Let's consider the domain $-\frac{\pi}{2} < \theta_1, \theta_2 < \frac{\pi}{2}$

($g(\theta_1, \theta_2) = 0$ would tell you how θ_2 changes according to θ_1)

Since $f(\theta_1, \theta_2) \to \infty$ as $\theta_1 \to \frac{\pi}{2}$

$f(\theta_1, \theta_2) \to \infty$ as $\theta_1 \to -\frac{\pi}{2}$

and there is only one critical point,

this must be an absolute minimum.